TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 340, Number 1, November 1993

REMOVING INDEX 0 FIXED POINTS FOR AREA PRESERVING MAPS OF TWO-MANIFOLDS

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ABSTRACT. Using the method of free modifications developed by M. Brown and extended to area preserving homeomorphisms, we prove the following fixed point removal theorem.

Theorem. *Let $h : M \to M$ be an orientation preserving, area preserving homeomorphism of an orientable two-manifold M having an isolated fixed point p of index* 0. *Given any open neighborhood N of p such that $N \cap \mathrm{Fix}(h) = p$, there exists an area preserving homeomorphism $\hat{h}$ such that*

(i) $\hat{h} = h$ *on* $\overline{M - N}$ *and*

(ii) $\hat{h}$ *is fixed point free on* N.

Two applications of this theorem are the second fixed point for the topological version of the Conley-Zehnder theorem on the two-torus, and a new proof of the second fixed point for the Poincaré-Birkhoff Fixed Point Theorem.

INTRODUCTION

Under what conditions can one remove, by local perturbation, a fixed point of a given homeomorphism of a manifold, or alternately, a stationary point for a continuous vector field? By "remove" we mean that the new map must be of the same type as the original (i.e., homeomorphism, smooth, area preserving, etc.). Clearly, the local fixed point index provides a necessary condition for the removal of the fixed point p; that is, if the index of p does not equal zero, then p cannot be removed. However, is this a sufficient condition?

For orientation preserving maps of orientable two manifolds, Schmitt [Sc] has shown sufficiency for homeomorphisms and Simon and Titus [ST] have shown sufficiency under the stronger hypothesis that the map is a C^k diffeomorphism which preserves area (however, only obtaining a C^1-local perturbation).

We prove sufficiency under the condition that h is an area preserving homeomorphism. The result of Schmitt is subsumed within this work.

Our strategy involves modifying both the homeomorphism and a simple closed curve (which surrounds the fixed point) to obtain a new homeomorphism and simple closed curve which is in standard position with respect to its image. In standard position one can perturb the homeomorphism so that it is both fixed point free and area preserving.

Received by the editors February 14, 1989 and, in revised form, October 30, 1990 and September 1, 1991.

1991 *Mathematics Subject Classification*. Primary 58F05.

Key words and phrases. Fixed point, area preserving, fixed point index.

The technique underlying these modifications is due to work of Brouwer [BW] concerning fixed point free homeomorphisms of the plane. The modifications themselves were constructed first by Brown [Br] for orientation preserving homeomorphisms of the plane, and later extended to area preserving homeomorphisms by Pelikan and Slaminka [PS].

We use the removal theorem to show the existence of fixed points to area preserving homeomorphisms of two manifolds. In Corollaries 3.1 and 3.2 we prove a topological version of the Conley-Zehnder Theorem (albeit generating only two fixed points) [C2] and present a new proof for the second fixed point of the Poincaré-Birkhoff theorem.

This paper is divided into three sections. In §1 we review the definition of fixed point index and define free modifications. In §2 we prove the removal theorem for area preserving, orientation preserving homeomorphisms of R^2 such that all of the fixed points are isolated and have index 0. In §3 we extend the results of §2 to orientable two manifolds and prove Corollaries 3.1 and 3.2.

1. Area preserving maps and the fixed point index

For completeness we define both the index of a simple closed curve and the fixed point index of an isolated fixed point.

Definition. Let $h: R^2 \to R^2$ be a homeomorphism of the plane, and C a simple closed curve such that $C \cap \mathrm{Fix}(h) = \varnothing$. The *index of* C *with respect to* h, denoted $\mathrm{ind}(h, C)$, is the degree of the map $H : C \to S^1$ given by $H(x) = (h(x) - x)/\|h(x) - x\|$, where $\| \ \|$ is the standard norm in R^2.

The two main properties of index which we need are:

(1) If h_t is an isotopy such that $C \cap \mathrm{Fix}(h_t) = \varnothing$ for all $0 \leq t \leq 1$, then $\mathrm{ind}(h_s, C) = \mathrm{ind}(h_t, C)$ for all $0 \leq s, t \leq 1$.

(2) If C_1, C_2 are two simple closed curves bounding an annulus A such that $A \cap \mathrm{Fix}(h) = \varnothing$, then $\mathrm{ind}(h, C_1) = \mathrm{ind}(h, C_2)$.

Due to the above properties, an isolated fixed point inherits an index.

Definition. Let $h : R^2 \to R^2$ be a homeomorphism of the plane and let $p \in \mathrm{Fix}(h)$ be an isolated fixed point. Let C be a simple closed curve bounding a disc D such that $p \in \mathrm{int}(D)$ and $D \cap \mathrm{Fix}(h) = p$. The *fixed point index of* p *with respect to* h, denoted by $\mathrm{ind}(h, p)$, is $\mathrm{ind}(h, C)$.

Since the fixed point index is clearly a local property, the above definition extends to isolated fixed points for homeomorphisms of two manifolds.

We now define the notion of a free modification.

Definition. Let $h : R^2 \to R^2$ be a homeomorphism of the plane and let $D \subset R^2$ be a disc such that $D \cap h(D) = \varnothing$. A *free modification* of h on D is the composition hg of h with a homeomorphism $g : R^2 \to R^2$ such that g is supported on D (i.e., $g(x) = x$ if $x \notin D$). We will also refer to a finite number of free modifications as a free modification.

Note that a free modification of h neither alters the fixed point set of h nor does it change the index of any simple closed curve and, thus, the index of any isolated fixed point.

The Brouwer Lemma [BW] is of central importance in our use of free modifications. Brown [Br1] has presented an extremely elegant proof of this lemma.

Brouwer Lemma. *Let $h : R^2 \to R^2$ be an orientation preserving homeomorphism of the plane and let $D \subset R^2$ be a topological disc such that $D \cap h(D) = \varnothing$. If $D \cap h^n(D) \neq \varnothing$ for some $n > 1$, then there exists a simple closed curve $C \subset R^2$ such that* $\operatorname{ind}(h, C) = 1$.

Brown [Br] has shown that if a homeomorphism lacks simple closed curves of index one, then the asymptotic behavior of each orbit of a point is easily understood.

Brown's Lemma [Br1, Lemma 3.4]. *Let $h : R^2 \to R^2$ be an orientation preserving homeomorphism of the plane having no simple closed curve of index one. Then for each $x \in R^2$,* $\limsup_{|x| \to \infty} h^n(x) \subset \operatorname{Fix}(h) \cup \infty$.

Even though a homeomorphism h may be area preserving, a free modification hf of h usually is not area preserving. However, if f is L-bi-Lipschitz (i.e., there exists an $L \geq 1$ such that $(1/L)|x-y| \leq |f(x)-f(y)| \leq L|x-y|$ for all x, $y \in R^2$) then hf does preserve a quasi-Lebesgue measure (cf. Pelikan and Slaminka [PS]).

Definition. A measure μ on a two-manifold is *quasi-Lebesgue* if it is absolutely continuous with respect to Lebesgue measure, zero on points and there exist positive constants K_1, K_2 such that $K_1 m(A) \leq \mu(A) \leq K_2 m(A)$ where A is a Lebesgue measurable set contained in some compact set and m is Lebesgue measure.

Note that Lebesgue measure is a quasi-Lebesgue measure.

We now cite an area preserving extension theorem due to Oxtoby and Ulam [OU].

Theorem [OU]. *Let $h : \operatorname{bd}(D) \to \operatorname{bd}(E)$ be an orientation preserving homeomorphism where D, E are topological discs in R^2 such that $\mu(D) = \mu(E)$ and $\mu(\operatorname{bd}(D)) = \mu(\operatorname{bd}(E)) = 0$ where μ is a quasi-Lebesgue measure. There exists a μ preserving homeomorphism $\hat{h} : D \to E$ such that $\hat{h} = h$ on* $\operatorname{bd}(D)$.

We will need the following lemma to construct one of our modifications. In this extension lemma we specify not only the map on the boundary of a disc but also on an arc in the interior.

Lemma 1.1. *Let D be a disc in R^2 such that $\mu(\operatorname{bd}(D)) = 0$ where μ is a quasi-Lebesgue measure, let $I \subset \operatorname{int}(D)$ be an arc with endpoints x, y such that $\mu(I) = 0$, and let $J \subset I$ be an arc with x as an endpoint. There exists an orientation preserving, μ preserving homeomorphism $f : D \to D$ such that*

(i) $f = \mathrm{id}$ *on* $\operatorname{bd}(D)$ *and*
(ii) $f(I) = J$.

Proof. Cut the disc into two discs such that I is on the boundary of each disc. Now specify homeomorphisms from each of the boundaries of the discs to themselves which are the identity except along the cut where the homeomorphisms map I onto J. Apply the Oxtoby-Ulam Theorem to each of these homeomorphisms and paste the discs together along the cut. □

The following proposition is a special case of one proved in Pelikan and Slaminka [PS] for free modifications.

Proposition 1.1. *Let h be an orientation preserving homeomorphism of R^2 which preserves a quasi-Lebesgue measure μ and let $\mathrm{Fix}(h)$ be isolated with $\mathrm{ind}(h, p) = 0$ for each $p \in \mathrm{Fix}(h)$. Suppose that hf is a free modification of h by a homeomorphism $f : R^2 \to R^2$ which is L-bi-Lipschitz. Then there exists a quasi-Lebesgue measure $\hat{\mu}$ preserved by hf.*

2. Removing index 0 fixed points in the plane

In this section we remove an index 0 isolated fixed point of an orientation preserving, area preserving homeomorphism of the plane.

Theorem 2.1. *Let $h : R^2 \to R^2$ be an orientation preserving, area preserving homeomorphism having a discrete fixed point set $\mathrm{Fix}(h)$. If each $p \in \mathrm{Fix}(h)$ has index 0, and N is an open neighborhood of some $p \in \mathrm{Fix}(h)$ such that $N \cap \mathrm{Fix}(h) = p$, then there exists an area preserving homeomorphism $\hat{h} : R^2 \to R^2$ such that*

(i) *$\hat{h} = h$ on $\overline{R^2 - N}$ and*
(ii) *$\hat{h}$ is fixed point free on N.*

Proof. Let C be a simple closed curve bounding a disc D such that:

(i) $p \in \mathrm{int}(D)$;
(ii) D, $h(D)$, $h^{-1}(D) \subset N$; and
(iii) $\mathrm{area}(C) = 0$.

Since $D \subset N$, it follows that $D \cap \mathrm{Fix}(h) = p$, thus $\mathrm{ind}(h, C) = 0$.

We first consider the general case where $h(C)$ intersects C either infinitely often or nontransversely.

Reduction to a finite number of transverse intersections. We claim that we can cover C with a finite number of discs E_i such that for each i:

(1) either $E_i \cap h(C)$ is connected and separates $\mathrm{int}(E_i)$ into two open discs F_{i0}, F_{i1}, or $E_i \cap h(C) = \varnothing$,

(2) $E_i \cap C$ is connected and separates $\mathrm{int}(E_i)$ into two open discs D_{i0}, D_{i1}, and

(3) $E_i \cap h^{-1}(E_i) = \varnothing$.

Since C is compact there exists a finite set of discs G_i covering C with properties (2) and (3). For each i, if G_i does not satisfy property (1), then, since $h(C)$ is locally connected, we can enlarge or shrink the disc G_i slightly so that $h(C)$ separates G_i into a finite number of components. Now divide each G_i into a finite union of discs satisfying properties (1) and (2). (See Figure 1.)

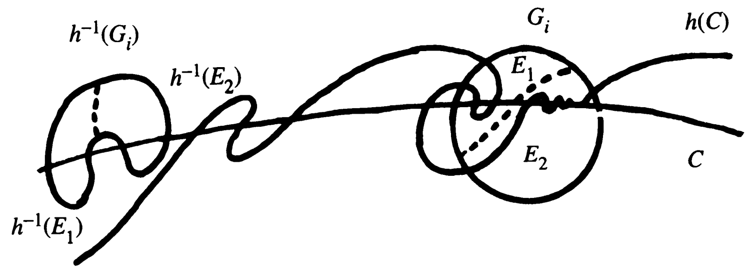

FIGURE 1

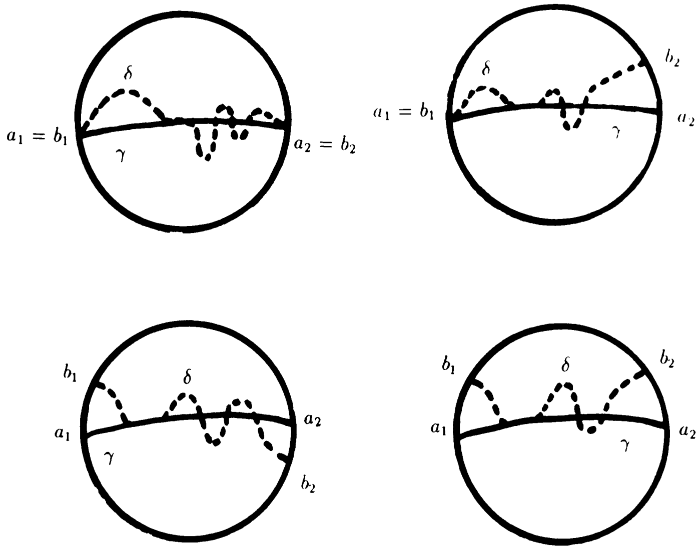

FIGURE 2

If a disc G_i was enlarged, one more subdivision may be needed to ensure that property (3) holds. Now let the set of the E_i's be this collection of discs.

Fix an E_i which we will now denote by E and for notational convenience make the following denotations: $\gamma = E \cap C$, $\delta = E \cap h(C)$, D_0, D_1 are the open discs in $\text{int}(E)$ separated by γ, and F_0, F_1 are the open discs in $\text{int}(E)$ separated by δ. Let a_1, a_2, be the endpoints of γ and let b_1, b_2 be the endpoints of δ. Finally let $f_0 = \overline{F_0} \cap \text{bd}(E)$ and similarly define f_1, d_0, and d_1. We have four possibilities for the position of a_1, a_2, b_1, b_2 on the boundary of E (Figure 2).

(1) $a_1 = b_1$ and $a_2 = b_2$ (or equivalently $a_1 = b_2$ and $a_2 = b_1$),
(2) $b_1 = a_1$ and $b_2 \in d_0 - \{a_1 \cup a_2\}$ (or the other equivalent cases),
(3) $b_1 \in d_0 - \{a_1 \cup a_2\}$ and $b_2 \in d_1 - \{a_1 \cup a_2\}$ (or the other equivalent case),
(4) b_1, $b_2 \in d_0 - \{a_1 \cup a_2\}$ or b_1, $b_2 \in d_1 - \{a_1 \cup a_2\}$.

In each of these cases let η be an arc in E, intersecting $\text{bd}(E)$ only at b_1

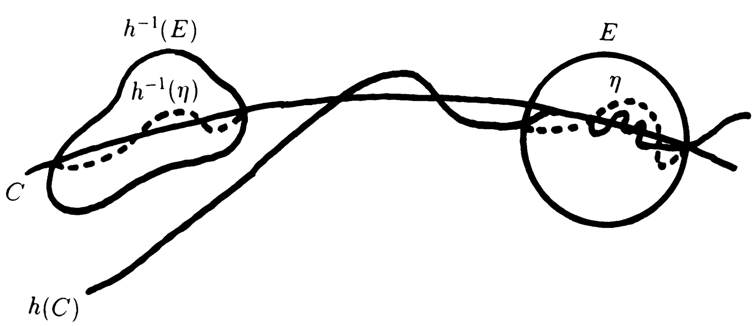

FIGURE 3

and b_2 such that η transversely intersects γ a finite number of times and such that the discs B_0, B_1 separated by η in E have the same area as F_0 and F_1 (determined by orientation). We now consider $h^{-1}(\eta)$ and apply the Oxtoby-Ulam Theorem to construct an area preserving homeomorphism g supported on $h^{-1}(E)$ which takes $h^{-1}(F_0)$ to $h^{-1}(B_0)$ and $h^{-1}(F_1)$ to $h^{-1}(B_1)$. The free modification hg thus has a finite number of transverse intersections in E (Figure 3).

Reduction to a connected component. By the above we can assume that $D \cap h(D)$ is the union of a finite number of components K_i, $i = 1, 2, \dots, n$, with the fixed point $p \in K_1$. We will construct a simple closed curve C_1 bounding a disc $D_1 \subset D$ with $p \in \operatorname{int}(D_1)$, and a modified homeomorphism h' such that $h'(D_1) \cap D_1$ has less than n components and such that h' preserves a quasi-Lebesgue measure. Thus repeating this procedure a finite number of times we can construct a simple closed curve C bounding a disc D and a modified quasi-Lebesgue measure preserving homeomorphism such that $D \cap h(D)$ is connected.

Let $\alpha \subset D \cap (\bigcup K_i)^c$ be an area 0 arc with endpoints a, b such that $\alpha \cap \operatorname{bd}(D \cap h(D)^c) = \{a, b\}$ and such that α separates K_1 from at least one other K_i in D (see Figure 4). Since $\alpha \cap h(\alpha) = \varnothing$ we may modify h, using the techniques in the previous reduction, so that $h(\alpha)$ intersects C at most finitely often.

Let $\beta_1, \beta_2 \subset C$ be the two arcs with endpoints a and b. If $h(\alpha) \cap K_i$ is connected for each i, let D_1 be the disc bounded by α and either β_1 or β_2 (whichever is such that $K_1 \subset D_1$). Then $D_1 \cap h(D_1)$ has less than n components.

Thus we may assume that $h(\alpha) \cap K_i$ is not connected for at least one i (see Figure 5). Let $x_1, x_2, \dots, x_m$ be the points of intersection of $h(\alpha)$ and C. Let the subscripts give an order to these points which is inherited from α, and let $x_i x_{i+1} \subset h(\alpha)$ denote the arc with endpoints x_i with x_{i+1}.

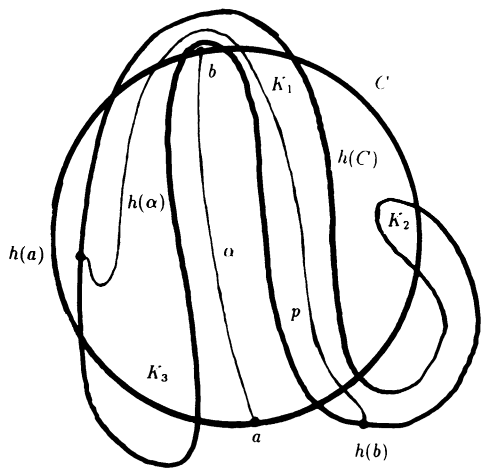

FIGURE 4

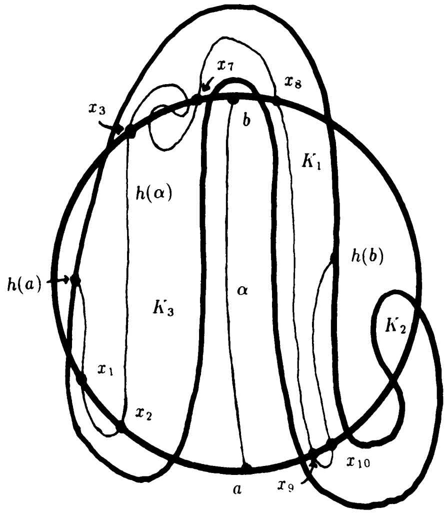

FIGURE 5

There exists an arc $x_i x_{i+1} \subset h(\alpha)$ such that x_i, $x_{i+1} \in \mathrm{bd}(K_j)$ for some j, and no other x_k lies between x_i, x_{i+1} on $\mathrm{bd}(K_j) \cap C$. For notational reasons, denote x_i by x, x_{i+1} by y, the arc $x_i x_{i+1}$ along $h(\alpha)$ by γ, and the component K_j by K. Either $\gamma \subset D$ or $\gamma \subset (\mathrm{int}(D))^c$. Let $\eta \in \mathrm{bd}(K) \cap C$ be the arc with endpoints x and y.

If $\gamma \subset (\mathrm{int}(h(D))^c$ then there exists a disc E containing $\eta \cup \gamma$ such that $E \cap h^{-1}(E) = \varnothing$ (see Figure 6). Let γ_1 be an arc in $K \cap E$ with endpoints on $h(\alpha)$ "near" x and y and intersecting $h(\alpha)$ only at these endpoints. Let $g : h^{-1}(E) \to h^{-1}(E)$ be an L-bi-Lipschitz homeomorphism which takes $\alpha \cap h^{-1}(E)$ to $h^{-1}(\gamma_1)$. The free modification hg now removes the intersection of $h(\alpha)$ at x and y.

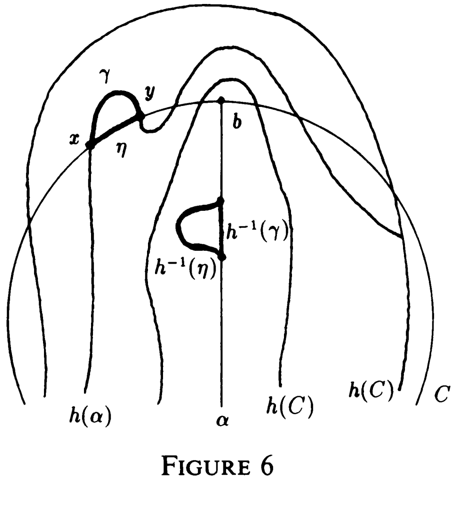

FIGURE 6

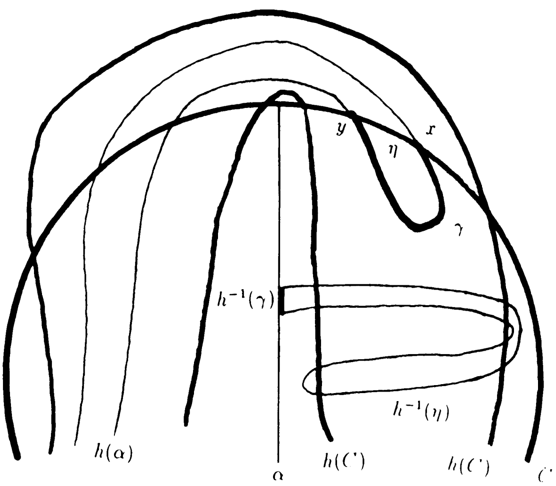

FIGURE 7

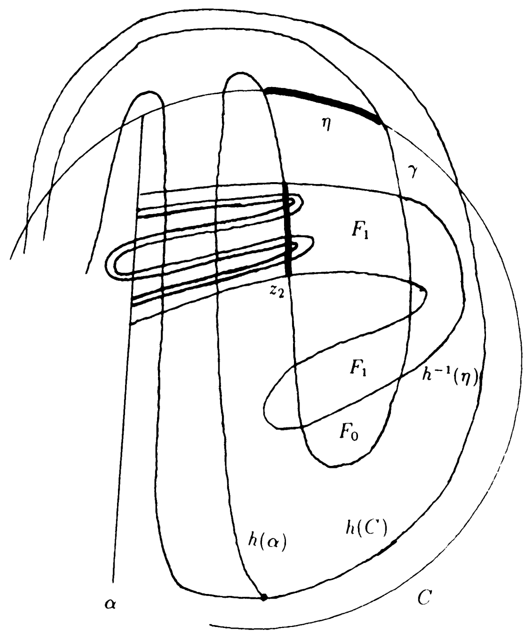

FIGURE 8

Thus we must now remove the intersection when $\gamma \subset D$ (see Figure 7). Without loss of generality we may assume that the fixed point is not contained in the disc bounded by $\eta \cup \gamma$. There are two possibilities: either $h^{-1}(\eta)$ intersects γ or it does not. If $h^{-1}(\eta) \cap \gamma = \varnothing$ then remove the intersection as above. So we now must assume that $h^{-1}(\eta) \cap \gamma \neq \varnothing$ (see Figure 8). Let F_0 denote the disc bounded by η and γ, let z_1 and z_2 be respectively the first and last points of intersection of γ and $h^{-1}(\eta)$, let η_1 denote the arc along $h^{-1}(\eta)$ from z_1 to z_2, let G_1 denote the disc bounded by $h^{-1}(\eta)$ and $h^{-1}(\gamma)$, and let γ_1 be the arc on γ from z_1 to z_2. Note that $p \notin F_1$. If G_1 is not a subset of F_0 then $h^{-1}(G_1 - F_0)$ is contained in the component of $D - \alpha$ which is not the same as the component D_1 containing F. Thus $(G_1 \cap F_0^c)$ does not intersect F_0.

Hence we need only consider $F_1 = G_1 \cap F_0$. Note that $h^{-1}(F_1) \subset h^{-1}(F_0)$. If F_1 is the empty set then we can merely apply the above modification to remove the intersection of γ with C. Else we repeat the last procedure on the closed set F_1 generating $F_2 \subset F_1$. If at any stage F_n is empty then we modify the homeomorphism to finally remove the intersection of γ with C. Thus, our only obstruction to this removal process is that for each n, F_n is nonempty. Let $z \in \bigcap F_n$. Then the positive orbit of z remains in F_0. But by Brown's Lemma this is possible only if the orbit of z converges to the fixed point, a contradiction. Thus at some stage the removal can be accomplished.

Perform the above removals (a finite number of times) until $h(\alpha) \cap K_i$ is connected for each i. We are now in the first case of the reduction to a connected component section.

Thus far we have both modified our homeomorphism and selected a simple closed curve (all performed within N) such that $h(C) \cap C$ consists of a finite number of points, $h(C)$ intersects C transversely, and $h(D) \cap D$ is connected. We now will modify h again so that C and $h(C)$ are in standard position from which the index information will allow us to prove the theorem.

Connected case. The set $h(C) \cap (\mathrm{int}(D))^c$ (see Figure 9) is a collection of arcs $\alpha_1, \dots, \alpha_n$ with endpoints a_i, b_i. Let $\beta_1, \dots, \beta_n$ be the arcs on C with endpoints a_i, b_i such that β_i separates $\alpha_i - \{a_i, b_i\}$ from p in $h(D)$.

We will partition the set $\{\alpha_1, \dots, \alpha_n\}$ into three sets (see Figure 10). *Overlap* will refer to those α_i such that $h^{-1}(\alpha_i)$ contains either a_i or b_i, but not both. *Disjoint* will refer to those α_i such that $h^{-1}(\alpha_i) \cap \beta_i = \varnothing$. *Hyperbolic/Elliptic* will refer to those α_i such that $h^{-1}(\alpha_i) \subset \beta_i$ (elliptic) or $h^{-1}(\alpha_i) \supset \beta_i$ (hyperbolic). Note that the case wherein $h^{-1}(\alpha_i)$ has endpoints in β_i but $h^{-1}(\alpha_i)$ is not a subset of β_i is not a possible configuration for a simple closed curve of index 0. If this were to occur then one sees easily that there is only one such arc α_1 with this property and the remaining arcs α_i are disjoint arcs. By a straightforward calculation $\mathrm{ind}(h, C) = 1$ (see Figure 11).

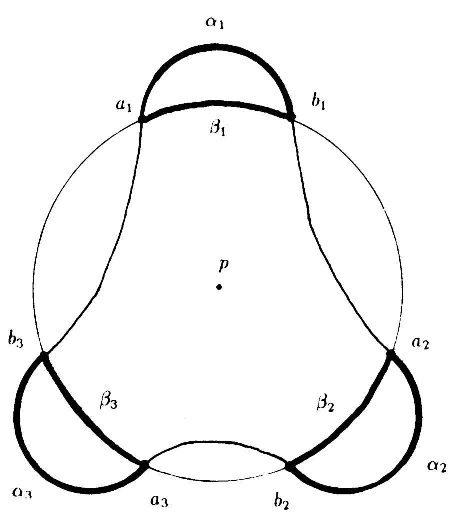

FIGURE 9

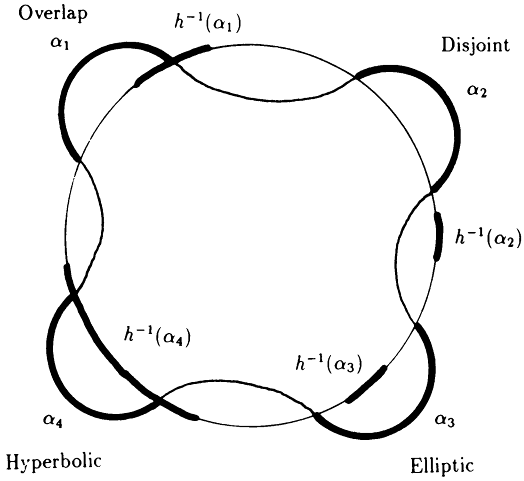

FIGURE 10

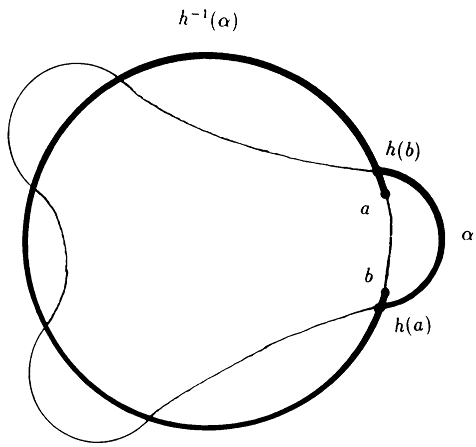

FIGURE 11

Our goal is to modify h to a quasi-Lebesgue measure preserving homeomorphism h' such that $h' = h$ on $\overline{R^2 - N}$ and $h'(C) \cap C$ consists of precisely two points (our standard position for index 0 simple closed curves). To accomplish this we will use Lemma 1.1 to transform the Overlap and Hyperbolic/Elliptic arcs into Disjoint arcs, and then modify h by an L-bi-Lipschitz homeomorphism to remove each of these Disjoint arcs. Our methods of proof will be based on induction on the number of points in $C \cap h(C)$, which is finite by assumption. We will first explain the modifications for each type of arc and then demonstrate the process in which we eliminate the number of points in $C \cap h(C)$.

Overlap Modification. Let α be an Overlap arc with endpoints a, b. Since h is orientation preserving, either a separates $h^{-1}(a)$ from $h^{-1}(b)$ along $h^{-1}(\alpha)$ in which case $h^{-1}(b) \in \beta$, or, b separates $h^{-1}(a)$ from $h^{-1}(b)$ along $h^{-1}(\alpha)$ in which case $h^{-1}(a) \in \beta$ (see Figure 12). Without loss of generality we may assume that a separates $h^{-1}(a)$ from $h^{-1}(b)$ along $h^{-1}(\alpha)$. Let $E \subset N$ be a disc such that the arc $ah^{-1}(b) \subset \operatorname{int}(E)$, where $ah^{-1}(b) \subset \beta$, and $E \cap h(E) = \varnothing$. Let c, $d \in E \cap (C - \beta)$ be points such that d separates c from a on $C \cap E$. By Lemma 1.1 there exists a homeomorphism g of R^2 onto itself, preserving the same measure which h preserves, which is the identity on $R^2 - E$ and maps the arc cd of $C \cap E$ onto the arc $ch^{-1}(b)$ of $C \cap E$. Now modify h by g to obtain a homeomorphism $hg : R^2 \to R^2$ which has the following properties:

(1) hg preserves the same measure as h;
(2) $hg = h$ on $R^2 - N$; and
(3) α is now a Disjoint arc.

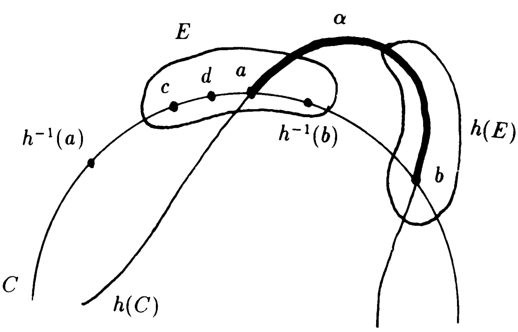

FIGURE 12

Hyperbolic/Elliptic Modification. Let α_1 be an elliptic arc with endpoints a_1, b_1 and let α_2 be a hyperbolic arc with endpoints a_2, b_2. Suppose that α_1 is "next to" α_2 in the sense that either the arc b_1a_2 or a_1b_2 along $h(C)$ (where b_1a_2 does not contain a_1 and a_1b_2 does not contain a_2) intersects C only at $\{b_1, a_2\}$ or $\{a_1, b_2\}$. Without loss of generality, we may assume the configuration of Figure 13. Let γ be the arc in $C \cap \overline{(D - h(D))}$ with endpoints a_2, b_1. By considering the map $h^{-1} : h(C) \to C$, we see that γ is an Overlap arc. By employing the Overlap Modification we can transform γ into a Disjoint arc.

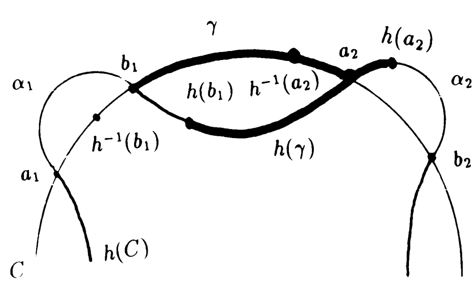

FIGURE 13

Disjoint Modification. In this case we will alter the homeomorphism and the measure that is preserved.

Lemma. *Let α be a Disjoint arc with endpoints a and b. Then $h^{-1}(\alpha \cup \beta) \cap (\alpha \cup \beta) = \varnothing$.*

Proof of lemma. Since $h^{-1}(\alpha) \cap \beta = \varnothing$ and $h^{-1}(\alpha) \subset C$, we have $h^{-1}(\alpha) \cap \alpha = \varnothing$. Clearly $h^{-1}(a \cup b) \cap \beta = \varnothing$ and since $\beta - (a \cup b) \subset \operatorname{int}(h(D))$, it follows that $h^{-1}(\beta - (a \cup b)) \subset \operatorname{int}(D)$. Thus $h^{-1}(\beta) \cap \beta = \varnothing$ and $h^{-1}(\beta) \cap \alpha = \varnothing$. □

Let E be a disc in N containing $\alpha \cup \beta$ such that $E \cap h^{-1}(E) = \varnothing$ (see Figure 14). This can be accomplished due to the above lemma. Let g be an L-bi-Lipschitz homeomorphism supported on $h^{-1}(E)$ such that $hg(h^{-1}(\beta - (a \cup b))) \subset \operatorname{int}(D)$.

The removal of the Disjoint arc has three consequences. First, it drops the number of intersections between C and $h(C)$ by 2, secondly, it may alter some of the other arcs, changing them from Disjoint to Overlap, or Overlap to Disjoint or Hyperbolic/Elliptic, etc., and, thirdly, we have altered the measure that is preserved.

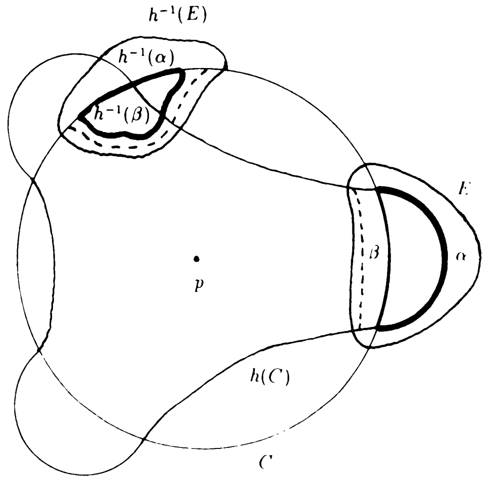

FIGURE 14

Reduction to standard position. We now use the above modifications to alter our homeomorphism and simple closed curve to one in standard position. Our simple closed curve C intersects $h(C)$ in $2n$ points. If there is a Disjoint arc, remove it using the Disjoint Modification. Continue until all of the Disjoint arcs have been removed (note, however, that although the application of a Disjoint Modification may change the number of other disjoint arcs, nevertheless the number of intersections between C and $h(C)$ will always decrease by 2). If any Overlap arcs exist, apply the Overlap Modification. This will produce a Disjoint arc, so the Disjoint Modification must be applied again until all are removed. After a finite number of modifications, we will be left with only hyperbolic and elliptic arcs. Apply the Hyperbolic/Elliptic modification. This creates an Overlap arc which can then be removed. Note again that we may have transformed some Hyperbolic or Elliptic arcs into Overlap arcs or Disjoint arcs, however, at each application of the Overlap Modification the number of intersection points of C and $h(C)$ is being reduced by 2. After a finite number of modifications we are left with all elliptic arcs, all hyperbolic arcs, or no arcs at all. It is easy to see that $\operatorname{ind}(h, C) = 1 + E - H$, where E and H are the number of elliptic and hyperbolic arcs. Since $\operatorname{ind}(h, C) = 0$, we must have one hyperbolic arc and two points of intersection.

Removal of fixed point. We now are able to remove the fixed point of index zero. For ease of notation let $g = hf_1 \cdots f_n$ and let μ be a quasi-Lebesgue measure preserved by g (from Proposition 1.1). We will construct a homeomorphism $\tau : D \to g(D)$ such that τ is fixed point free and preserves μ. Let α be the arc on C with endpoints a, b such that $\alpha \subset g(\operatorname{int}(D))^c$, let $\beta = \overline{C - \alpha}$, let γ be the arc on $g(C)$ with endpoints a, b such that $\gamma \subset D$, and let $\delta = \overline{g(C) - \gamma}$ (see Figure 15). Also denote the disc bounded by $\alpha \cup \gamma$ by E, the disc bounded by $\gamma \cup \beta$ by F, and the disc bounded by $\beta \cup \delta$ by G.

If $\mu(F) \leq \mu(G)$ then map F into G so that $\tau(\beta) = g(\beta)$ and τ preserves μ. Now map E into $\overline{G \cup F - \tau(F)}$ so that $\tau(\alpha) = g(\alpha)$ preserves μ. If $\mu(F) > \mu(G)$ pick a positive integer N such that $N > \mu(F)/\mu(G)$. Divide F into N discs $F_1, F_2, \dots, F_N$ such that the boundaries of F_i are arcs (of measure zero) from a to b and $\mu(F_i) = \mu(F_j)$ for all $1 \leq i,\ j \leq N$. Denote the boundary arcs of F_i by ϕ_{i1} and ϕ_{i2} where $\phi_{12} = \beta$ and $\phi_{N1} = \gamma$. Let τ map F_1 into G so that $\tau(\phi_{12}) = g(\beta)$ in a measure preserving fashion. Inductively for $i > 1$ let τ map F_i into $\overline{G \bigcup_{j=1}^{i-1} F_j - \bigcup_{j=1}^{i-1} \tau(F_j)}$ so that $\tau(\phi_{i2}) = \tau(\phi_{(i-1)1})$ and, as usual, τ preserves μ. Finally, let τ map E into $\overline{F \cup G - \bigcup_{j=1}^{N} \tau(F_j)}$ such that $\tau(\alpha) = g(\alpha)$.

Now, extend τ to R^2 by $\tau = g$ on $R^2 - D$. The homeomorphism τ is clearly seen to be fixed point free on D and τ preserves μ.

Now we have a homeomorphism which is fixed point free but unfortunately it does not agree with h on $\overline{R^2 - D}$ nor does it preserve area.

Let $\hat{h} = hg^{-1}\tau(x)$ be the "pullback" of τ. Note that $\hat{h} = h$ on $\overline{\operatorname{bd}(R^2 - D)}$. The map $\hat{h}$ also preserves area. We need only show that $\hat{h}$ is fixed point free on D. Since the only type of modification that altered the measure was a Disjoint Modification, we need only prove that $\hat{h}$ is fixed point free under the assumption that all of the modifications were of the Disjoint type. Without loss of generality, we shall assume that there was only one Disjoint Modification f on the disc $h^{-1}E$. Thus $\hat{h} = hf^{-1}h^{-1}\tau(x)$. So $\hat{h}$ has a fixed point if and only

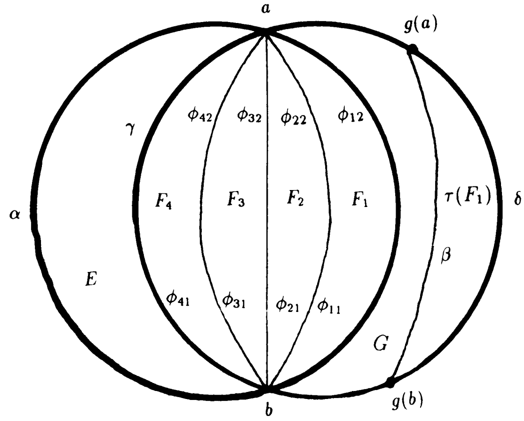

FIGURE 15

if $\tau(x) = hfh^{-1}(x)$. If $x \notin E$ then $h^{-1}(x) \notin h^{-1}(E)$ which implies that $\tau(x) = hfh^{-1}(x) = hh^{-1}(x) = x$, a contradiction. Thus the only way that $\hat{h}$ can have a fixed point is if $x \in E$. Since τ agrees with hf for $x \in C$ we see that $\hat{h}(x) = hf^{-1}h^{-1}\tau(x) = hf^{-1}h^{-1}hf(x) = h(x) \neq x$. If $x \in E \cap D^c$ then $\tau(x) = h(x)$. Thus $h(x) = \tau(x) = hfh^{-1}(x)$. But this implies that $x = fh^{-1}(x)$ which implies that $f^{-1}(x) = h^{-1}(x)$. However $x \notin h^{-1}(E)$ so $f^{-1}(x) = x$. Thus $x = h^{-1}(x)$ which is a contradiction. Thus the only way that $\hat{h}$ can have a fixed point is if $x \in E \cap D$. However $E \cap D$ can be chosen to be so "thin" that for every $x \in E \cap D$, $\tau(x) \approx h(x)$. Thus $\hat{h}$ is fixed point free. (This same argument applies to the reduction to the connected component case and, with h^{-1} in place of h, to the removal for the Hyperbolic/Elliptic Modification which are both disjoint modifications.) □

3. Main theorem and applications

Theorem 3.1. *Let $h : M^2 \to M^2$ be an area preserving, orientation preserving homeomorphism of an orientable two-manifold, having an isolated fixed point p of index* 0. *Given any open neighborhood N of p such that $N \cap \mathrm{Fix}(h) = p$, there exists an area preserving homeomorphism $\hat{h} : M^2 \to M^2$ such that*

(i) *$\hat{h} = h$ on $M^2 - N$ and*
(ii) *$\hat{h}$ is fixed point free on N.*

Proof. Let M_p be the connected component of $(M^2 - \mathrm{Fix}(h)) \cup \{p\}$ containing p. The map $h : M_p \to M_p$ is an area preserving, orientation preserving homeomorphism with a single fixed point p, and $N \subset M_p$.

Since $\mathrm{ind}(h, p) = 0$, by the Lefschetz fixed point theorem, the universal cover of M_p is R^2. Lift h to a homeomorphism $\tilde{h} : R^2 \to R^2$ such that $\tilde{h}$ has at least one fixed point $\tilde{p}$. Since the projection $\pi : R^2 \to M_p$ is a local homeomorphism, $\mathrm{ind}(\tilde{h}, \tilde{p}) = 0$. Pick a neighborhood $N_1 \subset N$ such that $\pi^{-1}(N_1)$ consists of disjoint copies homeomorphic to N_1. Denote the copy of N_1 containing $\tilde{p}$ by $\widetilde{N}_1$. Though $\tilde{h}$ does not necessarily preserve area, it does however preserve a measure which is absolutely continuous with respect to Lebesgue measure and Lebesgue measure is absolutely continuous with respect to it. We now apply Theorem 2.1 to eliminate the fixed point $\tilde{p}$. Since π is a homeomorphism on $\widetilde{N}_1$ we can push this new homeomorphism down to M_p and thus to M. □

Our first corollary is a topological version of the Conley-Zehnder theorem (though only generating two fixed points). The Conley-Zehnder theorem is a special case of the Arnol′d conjecture which concerns symplectic diffeomorphisms homologous to the identity. In the case of the two-torus, "homologous to the identity" is equivalent to the following definition (cf. [Ar, Appendix 9]).

Definition. Let $h: T^2 \to T^2$ be an area preserving, orientation preserving homeomorphism of a torus. Let $\tilde{h} : R^2 \to R^2$ be a lift of h and let D be a fundamental domain for $\tilde{h}$. The *mean translation vector of* $\tilde{h}$, denoted $\mathrm{mtv}(\tilde{h})$, is $\int_D (\tilde{h}(x) - x)\, dm$ where m is Lebesgue measure and the integral is a vector quantity.

We now show that our free modifications do not effect the mean translation vector.

Lemma 3.1. *Let $g : T^2 \to T^2$ be an area preserving, orientation preserving homeomorphism of the two-torus supported on a disc $E \subset T^2$, and let $\tilde{g} : R^2 \to R^2$ be a lift of f such that $\tilde{g}$ is supported on $\pi^{-1}(E)$ where π is the projection map $\pi : R^2 \to T^2$, then* $\operatorname{mtv}(\tilde{g}) = \vec{0}$.

Proof. Let $\widetilde{E}$ be a disc covering E and D a fundamental domain for $\tilde{g}$ such that $\widetilde{E} \subset D$. Define $\phi(x) = \tilde{g}(x) - x$ for $x \in R^2$. Note that $\phi(x) = \vec{0}$ for $x \in R^2 - \pi^{-1}(E)$ and $\tilde{g}$ is invariant on $\widetilde{E}$. Thus

$$\begin{aligned}\operatorname{mtv}(\tilde{g}) &= \int_D \phi(x)\, dm = \int_{\widetilde{E}} \phi(x)\, dm \\ &= \int_{\widetilde{E}} \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \phi(\tilde{g}^i(x_0))\, dm\end{aligned}$$

for almost all $x_0 \in \widetilde{E}$, by the Ergodic Theorem,

$$\begin{aligned}&= \int_{\widetilde{E}} \lim_{n\to\infty} \frac{1}{n} (\tilde{g}^{n+1}(x_0) - \tilde{g}(x_0))\, dm \\ &= \vec{0}\end{aligned}$$

since $\tilde{g}$ is invariant on $\widetilde{E}$. □

Since T^2 is compact π can be chosen to be area preserving on D. Thus any map covering an area preserving map is also area preserving.

Lemma 3.2. *If f, $h : T^2 \to T^2$ are orientation preserving, area preserving homeomorphisms of a two-torus with lifts $\tilde{f}$, $\tilde{h} : R^2 \to R^2$, then* $\operatorname{mtv}(\tilde{f}\tilde{h}) = \operatorname{mtv}(\tilde{f}) + \operatorname{mtv}(\tilde{h})$.

Proof. Let $\tilde{f}$, $\tilde{h}$ be lifts of T^2 to R^2 and let D be a fundamental domain. Then

$$\begin{aligned}\operatorname{mtv}(\tilde{f}\tilde{h}) &= \int_D (\tilde{f}\tilde{h}(x) - x)\, dm = \int_D (\tilde{f}\tilde{h}(x) - \tilde{h}(x))\, dm + \int_D (\tilde{h}(x) - x)\, dm \\ &= \int_{\tilde{h}(D)} (\tilde{f}(x) - x)\, dm\tilde{h}^{-1} + \operatorname{mtv}(\tilde{h}),\end{aligned}$$

where $m\tilde{h}^{-1}$ is the $\tilde{h}$-induced measure. However, since h is an area preserving homeomorphism, $dm\tilde{h}^{-1} = dm$ and thus

$$\int_{\tilde{h}(D)} (\tilde{f}(x) - x)\, dm\tilde{h}^{-1} = \int_D (\tilde{f}(x) - x)\, dm = \operatorname{mtv}(\tilde{f}). \quad \square$$

Observation. If f, $h : T^2 \to T^2$ are two orientation preserving, area preserving homeomorphisms of the two-torus such that $f = h$ outside a disc $E \subset T^2$ then $g = f^{-1}h : T^2 \to T^2$ is an area preserving homeomorphism which is supported on E. By Lemmas 3.1 and 3.2, there exist covering maps $\tilde{g}$, $\tilde{f}$, and $\tilde{h}$ such that $\operatorname{mtv}(\tilde{h}) = \operatorname{mtv}(\tilde{f}\tilde{g}) = \operatorname{mtv}(\tilde{f}) + \operatorname{mtv}(\tilde{g}) = \operatorname{mtv}(\tilde{f})$.

Corollary 3.1. *Let $h : T^2 \to T^2$ be an area preserving, orientation preserving homeomorphism of the two-torus which is homotopic to the identity and has a lift to R^2 with mean translation vector $\vec{0}$. Then h has at least two fixed points.*

Proof. By Franks [Fr], h possesses at least one fixed point p. Assume that h has no other fixed point. By the Lefschetz Index Theorem, $\operatorname{ind}(h, p) = 0$. Use Theorem 3.1 to remove this fixed point. Our modifications used to construct $\hat{h}$ involved either composing h with a homeomorphism g which was supported on a disc D or we replaced the homeomorphism by a homeomorphism $\hat{h}$ such that $\hat{h}^{-1}h$ was supported on a disc D. By Lemmas 3.1, 3.2, and the observation above, we have constructed an orientation preserving, area preserving homeomorphism $\hat{h} : T^2 \to T^2$ which is a fixed point free and has a lift with mean translation vector $\vec{0}$. Thus, there exist at least two fixed points. □

Our second corollary is a new proof for the second fixed point for the Poincaré-Birkhoff Theorem. The reader should contrast this proof with those of Birkhoff [B2], Carter [Ca], and Brown-Neumann [BN].

Corollary 3.2 (Poincaré-Birkhoff). *Let $h : A \to A$ be an orientation preserving, area preserving "twist" homeomorphism of the annulus. There exist at least two fixed points for h.*

Proof. By Franks [Fr] or Birkhoff [B1] there exists at least one fixed point p. If $\operatorname{ind}(h, p) = 0$ then, by Theorem 3.1, we can remove p giving us a contradiction. Thus $\operatorname{ind}(h, p) \neq 0$ which implies, by the Lefshetz Index Theorem, that there must exist at least two fixed points. □

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